Characteristic Formulae for Mechanized Program Verification

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Correctness as a theorem

– Using a proof assistant, one can state (virtually) any theorem and (possibly with some effort) build a proof of this theorem that the proof assistant can check.

– A statement of the form "this programs satisfies this specification" can be viewed as the statement of a theorem.

– So, in theory, proof assistants can be used to build machine-checked proofs of program correctness.

The question is: how?

Logical representation of programs

From a logical point of view, what is a program?

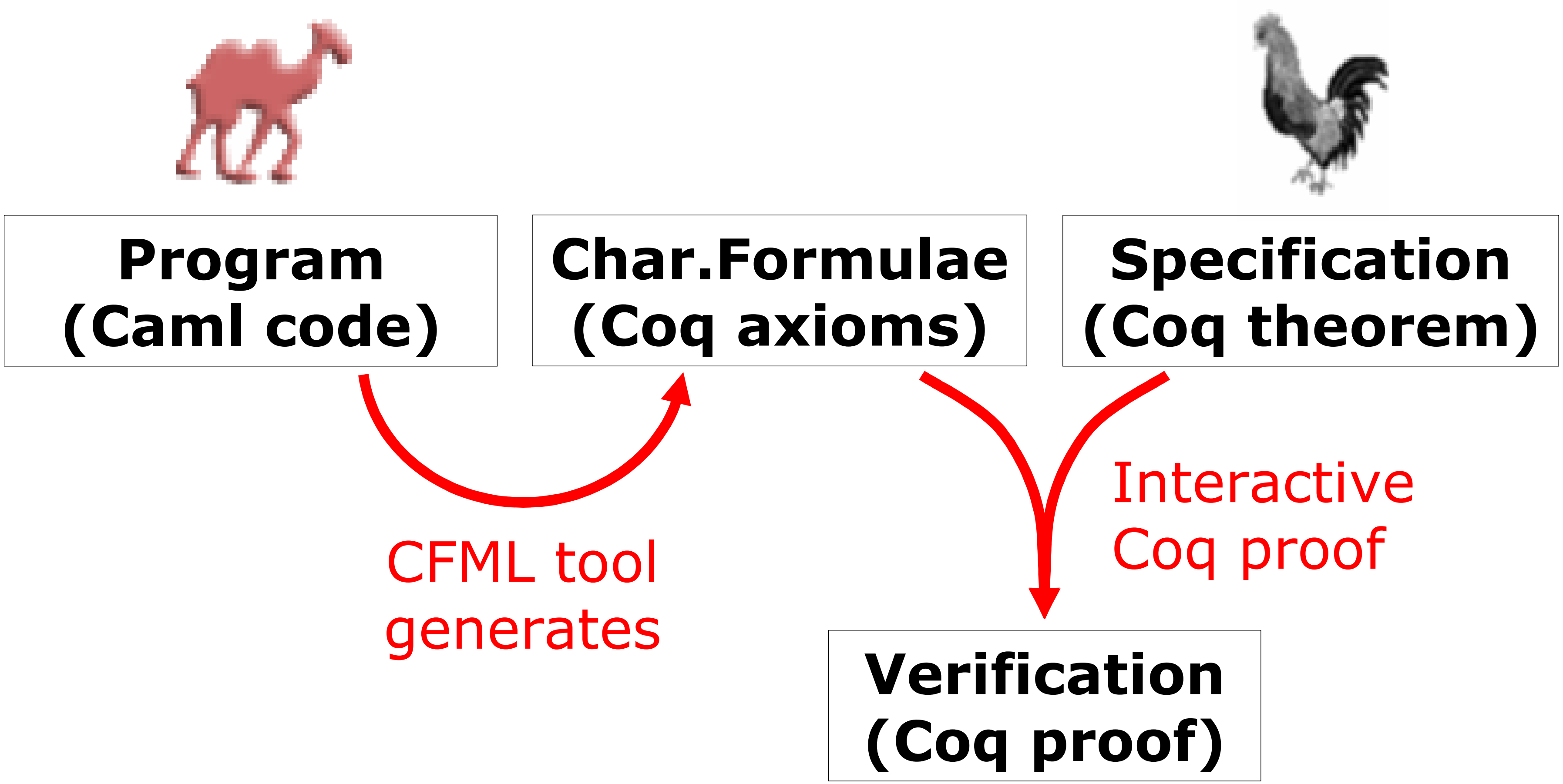
- syntactic view: a program is just a piece of syntax
- semantic view: a binary relation between states
- Hoare logic view: a relation between state predicates (possibly presented as a predicate transformer)

The three views can be formalized in the logic;
but we don't want to manipulate syntax explicitly;
moreover we prefer manipulating predicates directly,
as they allow for abstraction.

Remaining question: can we find an algorithm that produces, given an arbitrary program, the corresponding predicate transformer?

Characteristic formulae

I have developed an approach to program verification based on the generation of characteristic formulae (CF)



Program
(Caml code)

Char.Formulae
(Coq axioms)

Specification
(Coq theorem)

CFML tool
generates

Interactive
Coq proof

Verification
(Coq proof)

Closely related work

Origins of Characteristic Formulae:

– Hennessy-Milner logic (1980): two processes are bisimilar iff their characteristic formulae are equivalent

– Honda, Berger & Yoshida (2004,2006): one can build a most-general specification (i.e. Hoare triple) of any PCF program, without referring to a representation of syntax.

$\rightarrow$ strong logic: completeness with higher-order fcts

$\rightarrow$ yet ad-hoc logic, making it hard to reuse proof tools

Characteristic formulae in this work

1) **CF expressed in a standard higher-order logic**
→ accomodates a standard proof assistant

2) **CF with Separation Logic style specification**
→ supports modular verification

3) **CF of linear size and easy to read**
→ allows the approach to scale up

4) **Implementation of a CF generator**
→ supports verification of real Caml code

Proof assistants

User writes:

- definitions
- statement of theorems
- key steps of reasoning

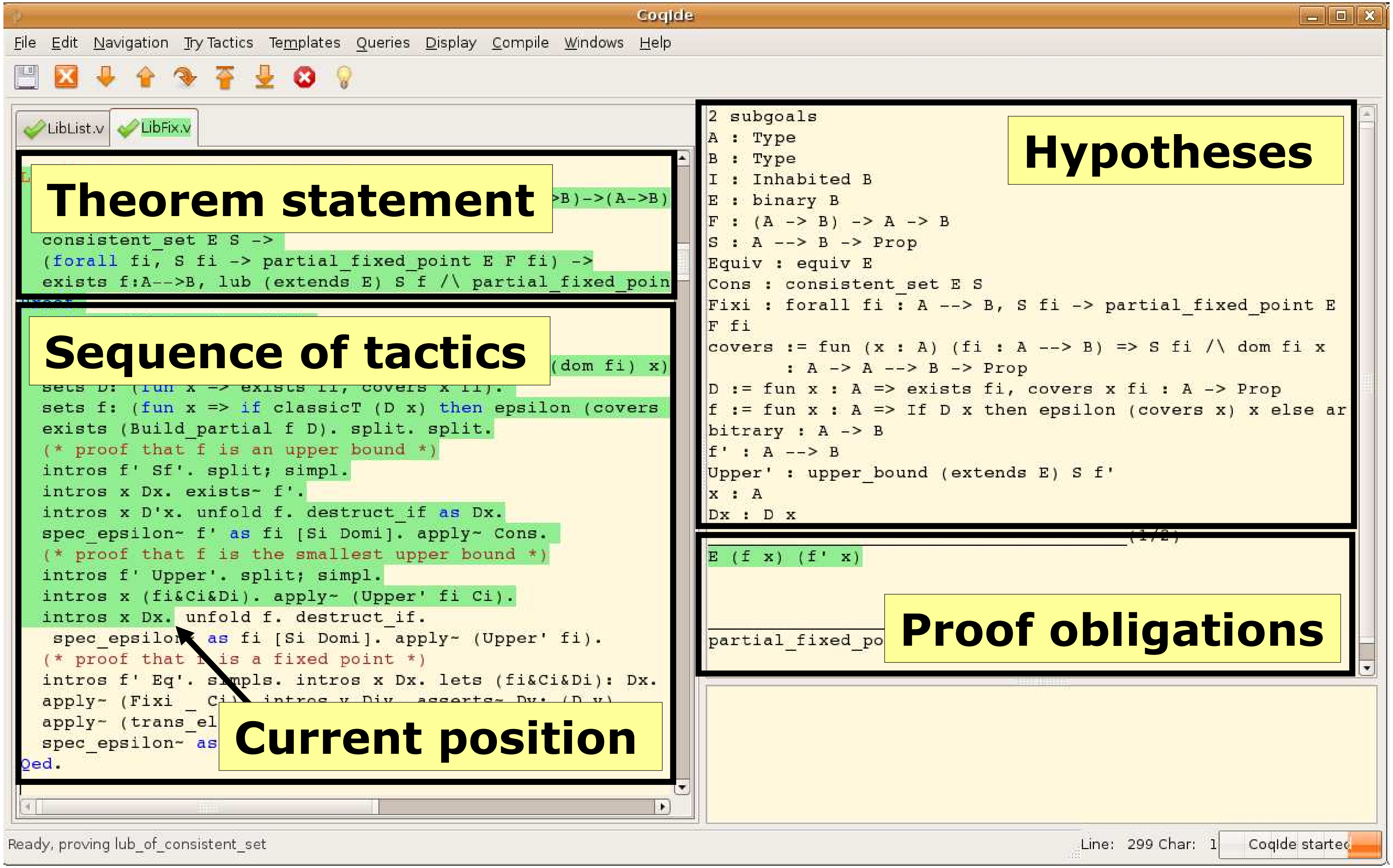
Proof assistant checks:

- well-formedness of definitions and statements
- legitimacy of each step of reasoning

No mistake possible:

If all the steps involved in the proof of theorem are accepted, then the theorem is true

Coq at a glance



```
>B)->(A->B)
  consistent_set E S ->
  (forall fi, S fi -> partial_fixed_point E F fi) ->
  exists f:A-->B, lub (extends E) S f /\ partial_fixed_poin
```

```
(dom fi) x)
  sets D: (fun x => exists fi, covers x fi).
  sets f: (fun x => if classicT (D x) then epsilon (covers
  exists (Build_partial f D). split. split.
  (* proof that f is an upper bound *)
  intros f' Sf'. split; simpl.
  intros x Dx. exists~ f'.
  intros x D'x. unfold f. destruct_if as Dx.
  spec_epsilon~ f' as fi [Si Domi]. apply~ Cons.
  (* proof that f is the smallest upper bound *)
  intros f' Upper'. split; simpl.
  intros x (fi&Ci&Di). apply~ (Upper' fi Ci).
  intros x Dx. unfold f. destruct_if.
   spec_epsilon~ as fi [Si Domi]. apply~ (Upper' fi).
  (* proof that f is a fixed point *)
  intros f' Eq'. simpls. intros x Dx. lets (fi&Ci&Di): Dx.
  apply~ (Fixi _ Ci). intros y Diy. asserts~ Dy: (D y).
  apply~ (trans_el
  spec_epsilon~ as
Qed.
```

```
2 subgoals
A : Type
B : Type
I : Inhabited B
E : binary B
F : (A -> B) -> A -> B
S : A --> B -> Prop
Equiv : equiv E
Cons : consistent_set E S
Fixi : forall fi : A --> B, S fi -> partial_fixed_point E
F fi
covers := fun (x : A) (fi : A --> B) => S fi /\ dom fi x
       : A -> A --> B -> Prop
D := fun x : A => exists fi, covers x fi : A -> Prop
f := fun x : A => If D x then epsilon (covers x) x else ar
bitrary : A -> B
f' : A --> B
Upper' : upper_bound (extends E) S f'
x : A
Dx : D x
______________________________________(1/2)
E (f x) (f' x)

______________________________________
partial_fixed_po
```

```
Ready, proving lub_of_consistent_set          Line:  299 Char:  1   CoqIde started
```

Specification

Heap *h*: finite map from locations to values

h : heap heap := fmap loc dyn

dyn := {A:Type; v:A}

Heap predicate *H*: description of a heap state

H : hprop hprop := heap → Prop

Hoare triple: ***{H} t {Q}*** asserts that, in an initial heap satisfying the predicate ***H***, the evaluation of the term ***t*** terminates and produces a value ***v*** such that the final heap satisfies the predicate **(*Q v*)**.

H is the *pre-condition* and *Q* is the *post-condition*

Characteristic formulae

The characteristic formula of a term t, written $[\![t]\!]$, is a higher-order predicate such that:

$$\forall H. \forall Q. \qquad [\![t]\!]\; H\; Q \qquad \Longleftrightarrow \qquad \{H\}\; t\; \{Q\}$$

→ obtain a predicate capturing the behavior of a program but not referring to the syntax of its code

→ translates source code into logical predicates

Note that $[\![t]\!]$ has type "hprop → (A → hprop) → Prop"

Soundness and completeness

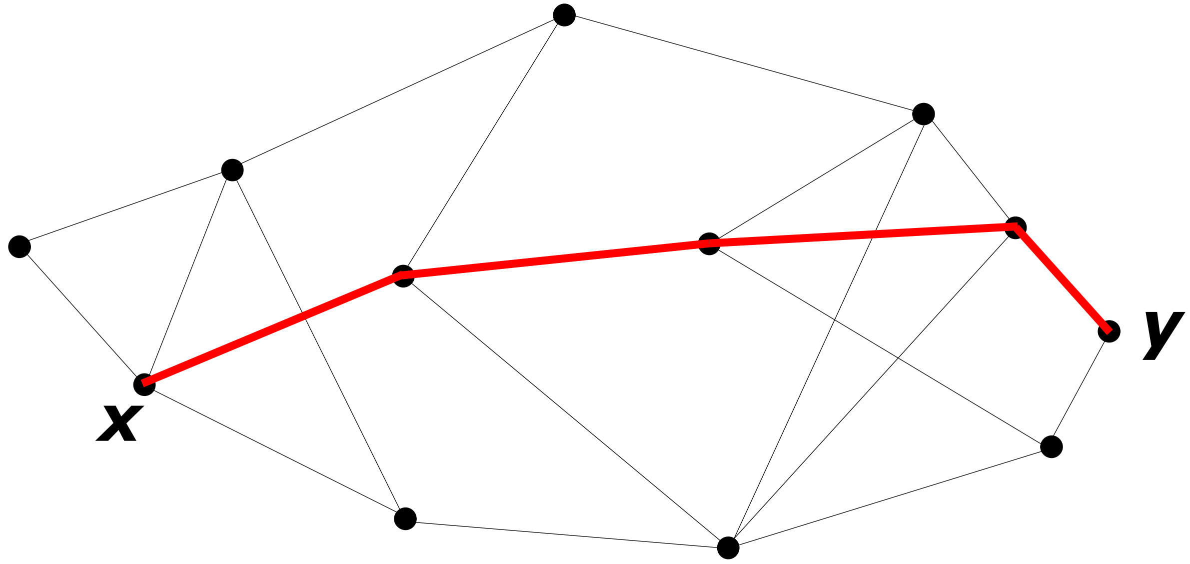
Soundness: if the CF of a program holds of a specification, then the program satisfies this spec.

$$\left\{\begin{array}{l} [\![t]\!]\, H\, Q \\ H\, h \end{array}\right. \quad \Rightarrow \quad \exists v.\, \exists h'.\, \left\{\begin{array}{l} t_{/h} \Downarrow v_{/h'} \\ Q\, v\, h' \end{array}\right.$$

Completeness: if a program satisifies a specification, then the CF of that program holds of that specification

$$t_{/\emptyset} \Downarrow n_{/h} \quad \Rightarrow \quad [\![t]\!]\ [\,]\ (\lambda x.\, [x = n])$$

Dijkstra's shortest path algorithm



v : bool array	marking of treated nodes
b : intbar array	storing best known distances
q : (int*int) pqueue	ordering the nodes to treat

where intbar = Finite of int | Infinite

Implementation

```
val dijkstra : ((int*int)list)array -> int -> int -> intbar
let dijkstra g s e =
   let n = Array.length g in
   let b = Array.make n Infinite in
   let v = Array.make n false in
   let q = Pqueue.create() in
   b.(s) <- Finite 0;
   Pqueue.push (s,0) q;
   while not (Pqueue.is_empty q) do
      let (x,dx) = Pqueue.pop q in
      if not v.(x) then begin
         v.(x) <- true;
         let update (y,w) =
           let dy = dx + w in
           if (match b.(y) with | Finite d -> dy < d
                                | Infinite -> true)
             then (b.(y) <- Finite dy; Pqueue.push (y,dy) q) in
         List.iter update g.(x);
      end;
   done;
   b.(e)
```

mutable structures

loop

pattern matching

higher-order function

abstract data structure

Material generated by CFML

```
Module Dijkstra (Pqueue : PqueueSig).

Axiom dijkstra : func.

Axiom dijkstra_cf :
   (@CFPrint.tag tag_top_fun _ _ (@CFPrint.tag tag_body _ _ (forall K :
   (CFHeaps.loc -> (int -> (int -> ((CFHeaps.hprop -> ((_ -> CFHeaps.hprop) ->
   Prop)) -> Prop)))), (                          : CFHeaps.loc, (forall s :
   int, (forall e : int,                          ag tag_let_trm (Label_create
   'n) _ (local (fun H                            (_ -> CFHeaps.hprop) =>
   (Logic.ex (fun Q1 : (int -> CFHeaps.hprop) => ((Logic.and (((@CFPrint.tag
   tag_apply _ _ ((((@app_1 CFHeaps.loc) int) ml_array_length)...
   (** goes on for about 100 more lines *)

End Dijkstra.
```

func = datatype used to represent functions

characteristic formula

→ The axioms are justified by the soundness theorem

Shortest path specification

```
Theorem dijkstra_spec : ∀ g x y G,
  nonnegative_edges G ->
  x \in nodes G ->
  y \in nodes G ->
  (App dijkstra g x y)
     (g ~> GraphAdjList G)
     (fun d => [d = dist G x y]
               \* g ~> GraphAdjList G)
```

mathematical graph

pre-condition

post-condition

→ Understanding the specification does not require particular skills with proof assistants

Main invariant

```
Definition hinv Q B V : hprop :=
   g ~> GraphAdjList G  (* G : graph int *)
\* v ~> Array V         (* V : array bool *)
\* b ~> Array B         (* B : array intbar *)
\* q ~> Pqueue Q        (* Q : multiset(int*int) *)
\* [inv Q B V].

Record inv Q B V : Prop := {
 Bdist: ∀x, x \in nodes G -> V\(x) = true ->
          B\(x) = dist G s x;
 Bbest: ∀x, x \in nodes G -> V\(x) = false ->
          B\(x) = mininf weight (crossing V x);
 Qcorr: ∀x, (x,d) \in Q ->
         x \in nodes G /\ ∃p, crossing V x p /\ weight p = d;
 Qcomp: ∀x p, x \in nodes G -> crossing V x p ->
          ∃d, (x,d) \in Q /\ d <= weight p;
 SizeV: length V = n;
 sizeB: length B = n }
```

Main lemma about invariant

```
Lemma inv_update : forall L V B Q x y
  x \in nodes G ->
  has_edge G x y w ->
  dy = dx + w ->
  Finite dx = dist G s x ->
  inv (V\(x:=true)) B Q (new_crossing
  If len_gt (B\(y)) dy
    then inv (V\(x:=true)) (B\(y:=Finite dy)) (\{(y, dy)} \u Q) ...
    else inv (V\(x:=true)) B Q (new_crossing x ((y,w)::L) V) .
Proof.
introv Nx Ed Dy Eq [Inv SV SB]. sets_
lets NegP: nonneg_edges_to_path Neg.
intros z. lets [Bd Bb Hc Hk]: Inv z.
(* case z = y *)
forwards~ (px&Px&Wx&Mx): (@mininf_fin
lets Ny: (has_edge_in_nodes_r Ed).
sets p: ((x,y,w)::px).
asserts W: (weight p = dy). subst p.
tests (V'\(y)) as C; case_If as Nlt.
(* subcase y visited, distance impro
false. rewrite~ Bd in Nlt. forwards M
 rewrite weight_cons in M. math.
(* subcase y visited, distance not improved *)
...
```

no reference to CF

maths-style reasoning in terms of multisets

All the nontrivial reasoning is there

180 lines of proofs in total for the invariant (a third in this lemma)

8 seconds to check

Verification of the code

```
Theorem dijkstra_spec : ∀ g x y G, ... (App dijkstra
Proof.
xcf. introv Pos Ns De. unfold GraphAdjList at 1. hdata_simpl.
xextract as N Neg Adj. xapp. intros Ln. rewrite <- Ln in Neg.
xapps. xapps. xapps. xapps*. xapps.
set (data := fun B V Q => g ~> Array N \*
  v ~> Array V \* b ~> Array B \* q ~> Heap Q).
set (hinv := fun VQ => let '(V,Q) := VQ in
 Hexists B, data B V Q \* [inv G n s V B Q (crossing
xseq (# Hexists V, hinv (V,\{})).
set (W := lexico2 (binary_map (count (= true)) (upto n))
                  (binary_map card (downto 0))).
xwhile_inv W hinv.
(* -- initial state satisfies the invariant -- *)
refine (ex_intro' (_,_)). unfold hinv,data. hsimpl.
 applys_eq~ inv_start 2. permut_simpl.
(* -- verification of the loop -- *)
intros [V Q]. unfold
(* ---- loop conditio
unfold data. xapps. x
(* ---- loop body --
...
Qed.
```

x-tactics

invariants

termination

lemma application

40 lines of proofs + 8 lines of invariants

15 seconds to check

Example of a proof obligation

```
Pos : nonnegative_edges G
Ns : s \in nodes G
Ne : e \in nodes G
Neg : nodes_index G n
Adj : forall x y w : int,
      x \in nodes G -> Mem (y, w) (N\(x)) = has_edge G x y w
Nx : x \in nodes G
Vx : ~ V\(x)
Dx : Finite dx = dist G s x
Inv : inv G n s V' B Q (new_crossing G s x L' V)
EQ : N\(x) = rev L' ++ (y, w) :: L
Ew : has_edge G x y w
Ny : y \in nodes G
______________________________________(1/6)
(Let dy := Ret dx + w in
  Let _x38 := App ml_array_get b y ; in
   If_ Match
       (Case _x38 = Finite d [d] Then Ret (dy '< d) Else
       (Case _x38 = Infinite Then Ret true Else Done))
   Then (App ml_array_set b y (Finite dy) ;) ;;
         App push (y, dy) h ; Else (Ret tt))
(q ~> Pqueue Q \* b ~> Array B \* v ~> Array V' \* g ~> Array N)
(fun _:unit => hinv' L)
```

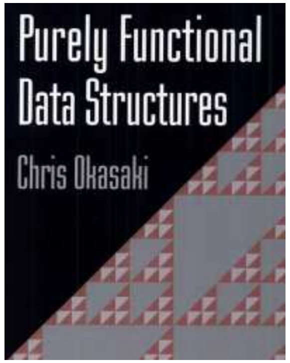
well-named hypotheses

char. formula

pre-condition

post-condition

Programs verified



Purely functional data structures: examples from Okasaki's book, including red-black trees, splay heaps, binomial heaps, pairing heaps, realtime queues, bootstrapped queues, random access lists

Imperative algorithms & data structures: dijsktra's shortest path, mutable lists, union-find, sparse arrays

Interaction between effects and functions:
- higher-order iterators on mutable structures (iter)
- closure with private local state (counter function)
- CPS functions (Reynold's CPS-append challenge)
- recursion through the store (Landin's knot)

Example of specification

$$t \quad = \quad \texttt{let x = }\underbrace{\texttt{!r + 1}}_{t_1}\texttt{ in }\underbrace{\texttt{s := x + 2}}_{t_2}$$

H = `(r ~~> 3) \* (s ~~> 9)`

Q′ = `fun v => [v = 4] \* (r ~~> 3) \* (s ~~> 9)`

The Hoare triple **{*H*}** t_1 **{*Q′*}** is true

Q′ x = `[x = 4] \* (r ~~> 3) \* (s ~~> 9)`

Q = `fun _:unit => (r ~~> 3) \* (s ~~> 6)`

The Hoare triple **{*Q′ x*}** t_2 **{*Q*}** is true

Thus, the Hoare triple **{*H*}** ***t*** **{*Q*}** is true

CF for let-expressions

Rule:

$$\frac{\{H\}\ t_1\ \{Q'\} \qquad \forall x.\ \{Q'\,x\}\ t_2\ \{Q\}}{\{H\}\ (\mathsf{let}\ x = t_1\ \mathsf{in}\ t_2)\ \{Q\}}$$

Goal: $\forall H. \forall Q. \quad [\![t]\!]\ H\ Q \quad \Longleftrightarrow \quad \{H\}\ t\ \{Q\}$

Definition:

$$[\![\mathsf{let}\ x = t_1\ \mathsf{in}\ t_2]\!] \ \equiv$$

$$\lambda H. \lambda Q.\ \ \exists Q'.\ [\![t_1]\!]\ H\ Q' \ \wedge\ \forall x.\ [\![t_2]\!]\ (Q'\,x)\ Q$$

Notation system for CF

CF for let-binding:

$$[\![\mathsf{let}\,x = t_1\,\mathsf{in}\,t_2]\!] \;\equiv$$

$$\lambda H.\,\lambda Q.\;\;\exists Q'.\;[\![t_1]\!]\,H\,Q' \;\wedge\; \forall x.\;[\![t_2]\!]\,(Q'\,x)\;Q$$

Definition of a Coq notation:

$$(\mathbf{Let}\;x = \mathcal{F}_1\;\mathbf{in}\;\mathcal{F}_2) \;\equiv$$

$$\lambda H.\,\lambda Q.\;\;\exists Q'.\;\mathcal{F}_1\,H\,Q' \;\wedge\; \forall x.\;\mathcal{F}_2\,(Q'\,x)\;Q$$

CF for let-binding, reformulated:

$$[\![\mathsf{let}\,x = t_1\,\mathsf{in}\,t_2]\!] \;\equiv\; (\mathbf{Let}\;x = [\![t_1]\!]\;\mathbf{in}\;[\![t_2]\!])$$

→ **translate a source code into a logical predicate**

Summary of CF generation

$$
\begin{array}{lcl}
\llbracket v \rrbracket & \equiv & \mathbf{Ret}\ v \\
\llbracket f\ v \rrbracket & \equiv & \mathbf{App}\ f\ v \\
\llbracket \mathsf{if}\ v\ \mathsf{then}\ t_1\ \mathsf{else}\ t_2 \rrbracket & \equiv & \mathbf{If}\ v\ \mathbf{then}\ \llbracket t_1 \rrbracket\ \mathbf{else}\ \llbracket t_2 \rrbracket \\
\llbracket \mathsf{let}\ x = t_1\ \mathsf{in}\ t_2 \rrbracket & \equiv & \mathbf{Let}\ x\ =\ \llbracket t_1 \rrbracket\ \mathbf{in}\ \llbracket t_2 \rrbracket \\
\llbracket \mathsf{let\ rec}\ f\ x = t_1\ \mathsf{in}\ t_2 \rrbracket & \equiv & \mathbf{Let\ rec}\ f\ x = \llbracket t_1 \rrbracket\ \mathbf{in}\ \llbracket t_2 \rrbracket \\
\llbracket \mathsf{crash} \rrbracket & \equiv & \mathbf{Crash} \\
\llbracket \mathsf{while}\ t_1\ \mathsf{do}\ t_2 \rrbracket & \equiv & \mathbf{While}\ \llbracket t_1 \rrbracket\ \mathbf{Do}\ \llbracket t_2 \rrbracket \\
\llbracket \mathsf{for}\ i = a\ \mathsf{to}\ b\ \mathsf{do}\ t \rrbracket & \equiv & \mathbf{For}\ i = a\ \mathbf{To}\ b\ \mathbf{Do}\ \llbracket t \rrbracket
\end{array}
$$

→ Characteristic formulae are easy to generate

→ Characteristic formulae are of linear size

→ Characteristic formulae read like source code

→ The user never needs to unfold the definitions

Conclusion

- **A new, practical approach** to program verification

- **Soundness** and **completeness** proofs

- **Implementation**: CFML, from Caml to Coq

- **Examples**: verification can be achieved at fairly reasonable cost even for complex algorithms

The end!

Further information and examples: http://arthur.chargueraud.org/