Characteristic Formulae for Mechanized Program Verification

Arthur Charguéraud

Max Planck Institute for Software Systems

INRIA Rennes

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Big programs everywhere

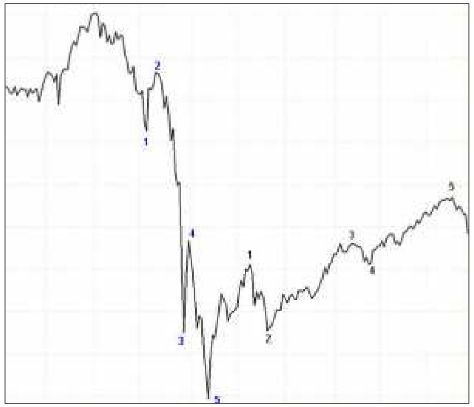
Programs are everywhere

Programs are ever-more complex

→ 10 million lines of code in your pocket

What if one of those lines was incorrect?

Cell phones are not the only devices that may crash...



Bugs everywhere

If suffices to have one single line incorrect to end up with a buggy system. How can we prevent that?

1) Code review

→ extremely hard for humans to catch all bugs

2) Test

→ find some bugs, but others remain undetected

3) Static analysis (e.g. type checking)

→ find all the bugs of a particular kind

4) Mechanized verification

→ use a machine to prove the absence of bug

Specification

Definition: a specification is a description of what a program is intended to compute, regardless of how the program computes its result

Examples of specifications:

– the definition **let n = ...** produces a value ***n*** that is the smallest prime number greater than 90

– the function **let f x = ...**, when given a nonnegative integer **x**, returns an integer equal to **x!**

– the function **let incr r = ...**, when called in a state where the location **r** contains an integer **n**, changes the memory so that the location **r** contains **n+1**

Correctness as a theorem

The statement "this program is free of bug" can be formulated as a formal theorem:

"This program admits that specification"

→ In general, we cannot expect a machine to automatically prove theorems of this form

→ Some form of human intervention is needed

→ One possibility is to use a **proof assistant** (e.g., Coq, Isabelle, HOL4, ...)

Proof assistants

User writes:

– definitions

– statement of theorems

– key steps of reasoning

Proof assistant checks:

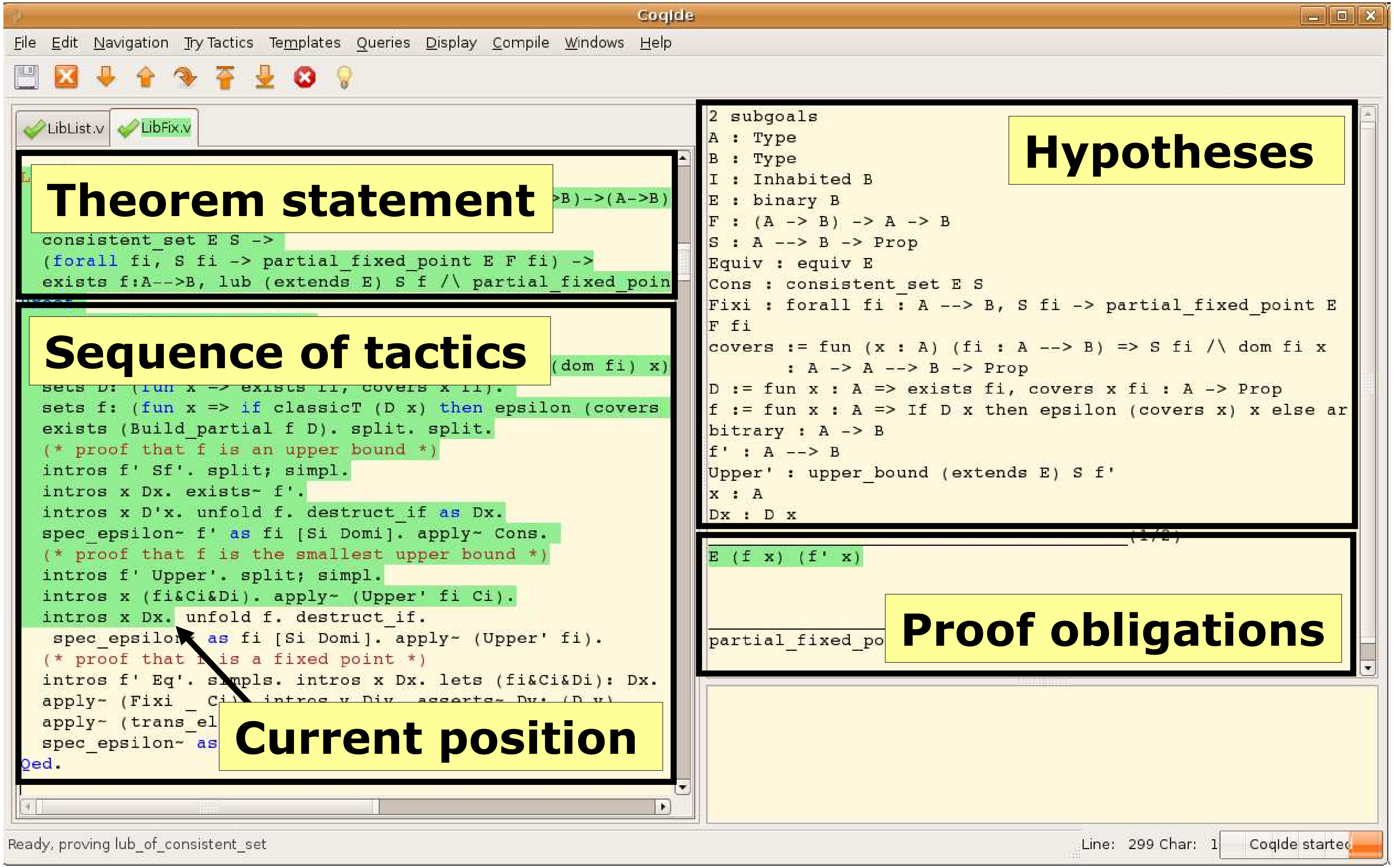
– well-formedness of definitions and statements

– legitimacy of each step of reasoning

No mistake possible:

If all the steps involved in the proof of theorem are accepted, then the theorem is true

Coq at a glance



```
>B)->(A->B)
consistent_set E S ->
(forall fi, S fi -> partial_fixed_point E F fi) ->
exists f:A-->B, lub (extends E) S f /\ partial_fixed_poin
```

```
(dom fi) x)
sets D: (fun x => exists fi, covers x fi).
sets f: (fun x => if classicT (D x) then epsilon (covers
exists (Build_partial f D). split. split.
(* proof that f is an upper bound *)
intros f' Sf'. split; simpl.
intros x Dx. exists~ f'.
intros x D'x. unfold f. destruct_if as Dx.
spec_epsilon~ f' as fi [Si Domi]. apply~ Cons.
(* proof that f is the smallest upper bound *)
intros f' Upper'. split; simpl.
intros x (fi&Ci&Di). apply~ (Upper' fi Ci).
intros x Dx. unfold f. destruct_if.
 spec_epsilon~ as fi [Si Domi]. apply~ (Upper' fi).
(* proof that f is a fixed point *)
intros f' Eq'. simpls. intros x Dx. lets (fi&Ci&Di): Dx.
apply~ (Fixi _ Ci).
apply~ (trans_el
spec_epsilon~ as
Qed.
```

```
2 subgoals
A : Type
B : Type
I : Inhabited B
E : binary B
F : (A -> B) -> A -> B
S : A --> B -> Prop
Equiv : equiv E
Cons : consistent_set E S
Fixi : forall fi : A --> B, S fi -> partial_fixed_point E
F fi
covers := fun (x : A) (fi : A --> B) => S fi /\ dom fi x
    : A -> A --> B -> Prop
D := fun x : A => exists fi, covers x fi : A -> Prop
f := fun x : A => If D x then epsilon (covers x) x else ar
bitrary : A -> B
f' : A --> B
Upper' : upper_bound (extends E) S f'
x : A
Dx : D x
______________________________________(1/2)
E (f x) (f' x)

partial_fixed_po
```

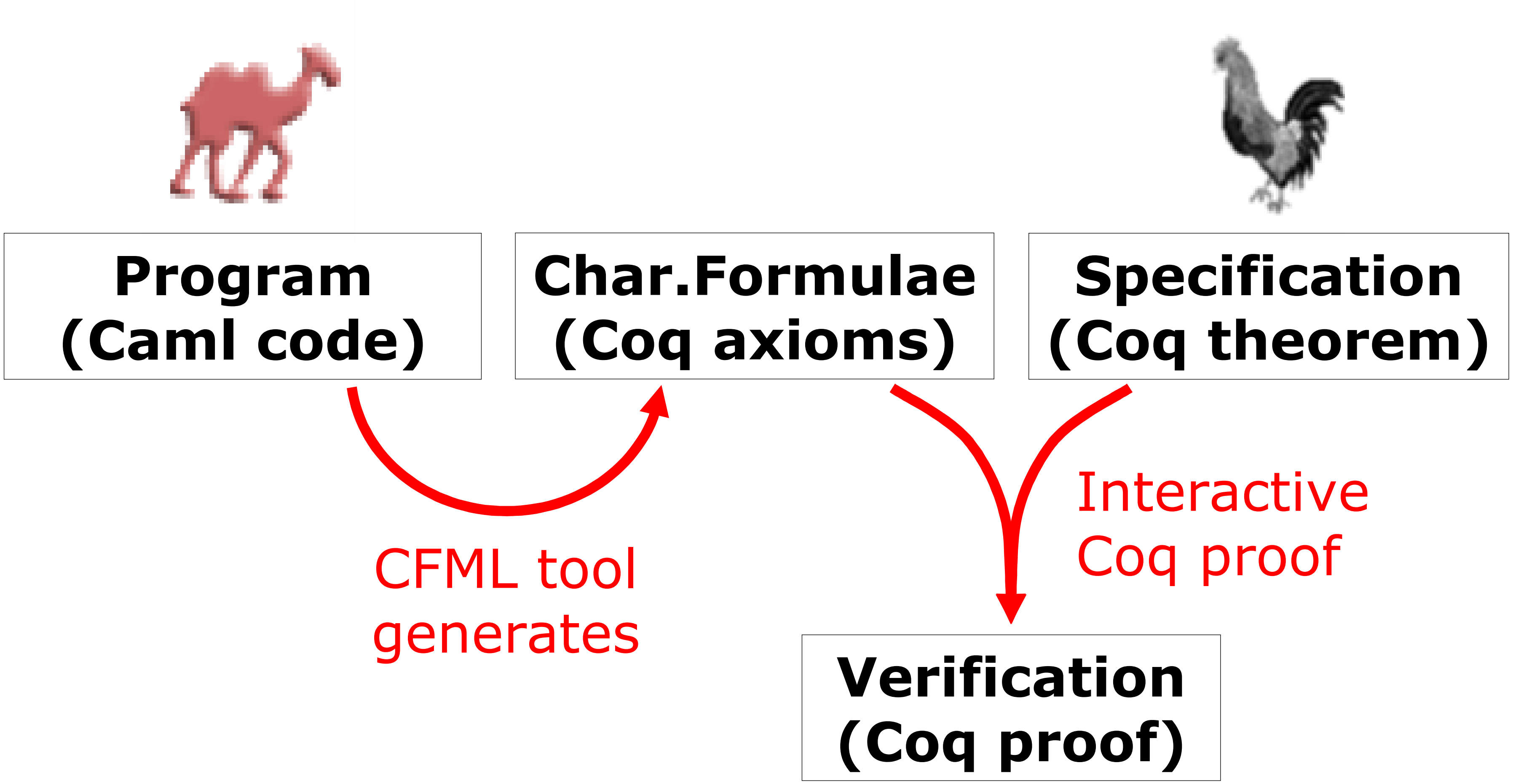
Ready, proving lub_of_consistent_set

Line: 299 Char: 1

CoqIde started

Characteristic formulae

In this thesis: a new, practical approach to program verification based on **Characteristic Formulae (CF)**



Program (Caml code)

Char.Formulae (Coq axioms)

Specification (Coq theorem)

CFML tool generates

Interactive Coq proof

Verification (Coq proof)

- **Introduction**
- **Theory: construction of CF**
  - specification language
  - description of values in Coq
  - CF for let-bindings
  - notation system for CF
  - soundness and completeness
- **Practice: Dijkstra's algorithm**
- **CF in the design space**
- **Conclusion**

Specification

Heap *h*: finite map from locations to values

h : heap heap := fmap loc dyn

dyn := {A:Type; v:A}

Heap predicate *H*: description of a heap state

H : hprop hprop := heap → Prop

Hoare triple: **{*H*} *t* {*Q*}** asserts that, in an initial heap satisfying the predicate ***H***, the evaluation of the term ***t*** terminates and produces a value ***v*** such that the final heap satisfies the predicate **(*Q v*)**.

H is the *pre-condition* and *Q* is the *post-condition*

Example of specification

$$t \quad = \quad \texttt{let x = }\underbrace{\texttt{!r + 1}}_{t_1}\texttt{ in }\underbrace{\texttt{s := x + 2}}_{t_2}$$

H = `(r ~~> 3) \* (s ~~> 9)`

Q' = `fun v => [v = 4] \* (r ~~> 3) \* (s ~~> 9)`

The Hoare triple **{*H*} *t₁* {*Q'*}** is true

$Q'\,x$ = `[x = 4] \* (r ~~> 3) \* (s ~~> 9)`

Q = `fun _:unit => (r ~~> 3) \* (s ~~> 6)`

The Hoare triple **{*Q' x*} *t₂* {*Q*}** is true

Thus, the Hoare triple **{*H*} *t* {*Q*}** is true

Representation of values

Caml values are represented as Coq values

– Base values are translated directly: a Caml value of type ***bool list*** becomes a Coq value of type **list bool**

– A Caml reference of type ***T ref*** is described in Coq as a value of type **loc** (`r` has type `loc` in `r ~~> 3`)

– A Caml function of type $T_1 \rightarrow T_2$ is described in Coq as a value of an abstract type called **func**, and it is specified with help of an abstract predicate called **App**

Note: for simplicity, the type "int" is mapped to "Z"

Characteristic formulae

The characteristic formula of a term t, written $[\![t]\!]$, is a higher-order predicate such that:

$$\forall H. \forall Q. \qquad [\![t]\!]\; H\; Q \qquad \Longleftrightarrow \qquad \{H\}\; t\; \{Q\}$$

→ obtain a predicate capturing the behavior of a program but not referring to the syntax of its code

→ **translates source code into logical predicates**

Note that $[\![t]\!]$ has type "hprop → (A → hprop) → Prop"

CF for let-expressions

Rule:

$$\frac{\{H\}\ t_1\ \{Q'\} \qquad \forall x.\ \{Q'\ x\}\ t_2\ \{Q\}}{\{H\}\ (\mathsf{let}\ x = t_1\ \mathsf{in}\ t_2)\ \{Q\}}$$

Goal: $\forall H. \forall Q. \quad [\![t]\!]\ H\ Q \iff \{H\}\ t\ \{Q\}$

Definition:

$$[\![\mathsf{let}\ x = t_1\ \mathsf{in}\ t_2]\!] \equiv$$

$$\lambda H. \lambda Q.\ \ \exists Q'.\ [\![t_1]\!]\ H\ Q' \ \wedge\ \forall x.\ [\![t_2]\!]\ (Q'\ x)\ Q$$

Notation system for CF

CF for let-binding:

$$[\![\mathsf{let}\, x = t_1 \,\mathsf{in}\, t_2]\!] \;\equiv$$

$$\lambda H.\, \lambda Q.\;\; \exists Q'.\; [\![t_1]\!]\, H\, Q' \;\;\wedge\;\; \forall x.\; [\![t_2]\!]\, (Q'\, x)\; Q$$

Definition of a Coq notation:

$$(\mathbf{Let}\; x = \mathcal{F}_1 \;\mathbf{in}\; \mathcal{F}_2) \;\equiv$$

$$\lambda H.\, \lambda Q.\;\; \exists Q'.\; \mathcal{F}_1\, H\, Q' \;\;\wedge\;\; \forall x.\; \mathcal{F}_2\, (Q'\, x)\; Q$$

CF for let-binding, reformulated:

$$[\![\mathsf{let}\, x = t_1 \,\mathsf{in}\, t_2]\!] \;\equiv\; (\mathbf{Let}\; x = [\![t_1]\!] \;\mathbf{in}\; [\![t_2]\!])$$

→ **translate a source code into a logical predicate**

Summary of CF generation

$$\begin{array}{lcl}
[\![v]\!] & \equiv & \mathbf{Ret}\ v \\
[\![f\ v]\!] & \equiv & \mathbf{App}\ f\ v \\
[\![\text{if } v \text{ then } t_1 \text{ else } t_2]\!] & \equiv & \mathbf{If}\ v\ \mathbf{then}\ [\![t_1]\!]\ \mathbf{else}\ [\![t_2]\!] \\
[\![\text{let } x = t_1 \text{ in } t_2]\!] & \equiv & \mathbf{Let}\ x\ =\ [\![t_1]\!]\ \mathbf{in}\ [\![t_2]\!] \\
[\![\text{let rec } f\ x = t_1 \text{ in } t_2]\!] & \equiv & \mathbf{Let\ rec}\ f\ x = [\![t_1]\!]\ \mathbf{in}\ [\![t_2]\!] \\
[\![\text{crash}]\!] & \equiv & \mathbf{Crash} \\
[\![\text{while } t_1 \text{ do } t_2]\!] & \equiv & \mathbf{While}\ [\![t_1]\!]\ \mathbf{Do}\ [\![t_2]\!] \\
[\![\text{for } i = a \text{ to } b \text{ do } t]\!] & \equiv & \mathbf{For}\ i = a\ \mathbf{To}\ b\ \mathbf{Do}\ [\![t]\!]
\end{array}$$

→ Characteristic formulae are easy to generate

→ Characteristic formulae are of linear size

→ Characteristic formulae read like source code

→ The user never needs to unfold the definitions

Soundness and completeness

Soundness: if the CF of a program holds of a specification, then the program satisfies this spec.

$$\left\{\begin{array}{l} [\![t]\!]\; H\; Q \\ H\; h \end{array}\right. \quad\Rightarrow\quad \exists v.\, \exists h'.\; \left\{\begin{array}{l} t_{/h} \Downarrow v_{/h'} \\ Q\; v\; h' \end{array}\right.$$

Completeness: if a program satisifies a specification, then the CF of that program holds of that specification

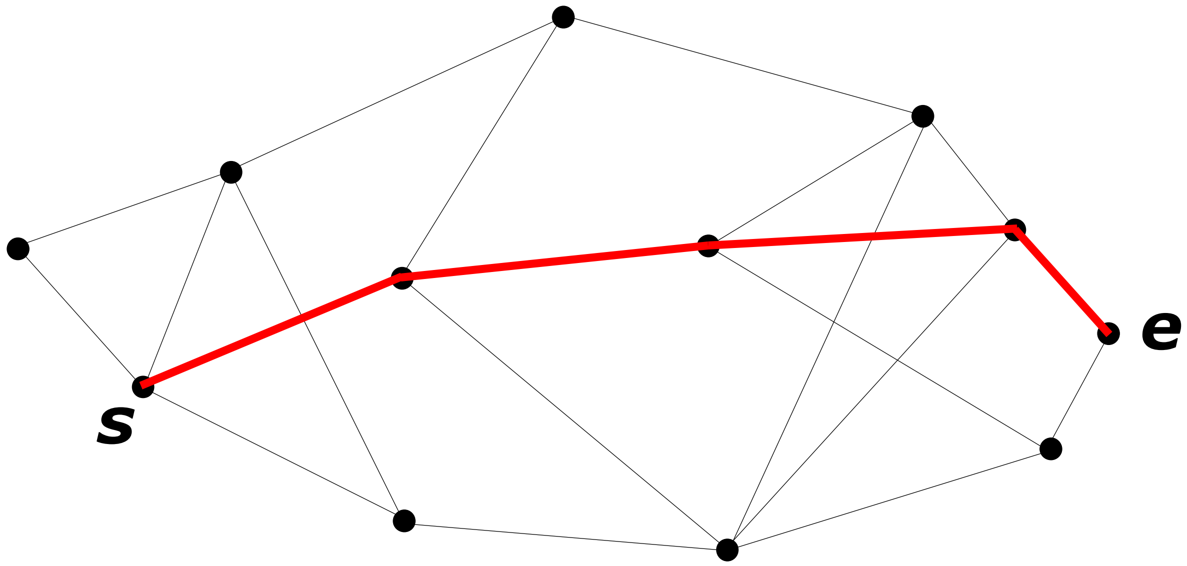
$$t_{/\emptyset} \Downarrow n_{/h} \quad\Rightarrow\quad [\![t]\!]\; [\,]\; (\lambda x.\, [x = n])$$

Meaning: characteristic formulae tell all the truth, and nothing but the truth, about the behavior of a program

- Introduction
- Theory: construction of CF
- **Practice: Dijkstra's algorithm**
  - overview of the source code
  - material generated by CFML
  - specification and invariants
  - overview of the proof scripts
  - other examples formalized
- CF in the design space
- Conclusion

Dijkstra's shortest path algorithm

Path of minimum weight from a node ***s*** to a node ***e***



v : bool array	marking of treated nodes
b : intbar array	storing best known distances
q : (int*int) pqueue	ordering the nodes to treat

where intbar = Finite of int | Infinite

Implementation

```
val dijkstra : ((int*int)list)array -> int -> int -> intbar
let dijkstra g s e =
   let n = Array.length g in
   let b = Array.make n Infinite in
   let v = Array.make n false in
   let q = Pqueue.create() in
   b.(s) <- Finite 0;
   Pqueue.push (s,0) q;
   while not (Pqueue.is_empty q) do
      let (x,dx) = Pqueue.pop q in
      if not v.(x) then begin
         v.(x) <- true;
         let update (y,w) =
           let dy = dx + w in
           if (match b.(y) with | Finite d -> dy < d
                                | Infinite -> true)
             then (b.(y) <- Finite dy; Pqueue.push (y,dy) q) in
         List.iter update g.(x);
      end;
   done;
   b.(e)
```

mutable structures

loop

pattern matching

higher-order function

abstract data structure

Material generated by CFML

```
Module Dijkstra (Pqueue : PqueueSig).

Axiom dijkstra : func.
```

func = datatype used to represent functions

```
Axiom dijkstra_cf :
    (@CFPrint.tag tag_top_fun _ _ (@CFPrint.tag tag_body _ _ (forall K :
    (CFHeaps.loc -> (int -> (int -> ((CFHeaps.hprop -> ((_ -> CFHeaps.hprop) ->
    Prop)) -> Prop)))), (                            ) : CFHeaps.loc, (forall s :
    int, (forall e : int,                            ag tag_let_trm (Label_create
    'n) _ (local (fun H :                            (_ -> CFHeaps.hprop) =>
    (Logic.ex (fun Q1 : (int -> CFHeaps.hprop) => ((Logic.and (((@CFPrint.tag
    tag_apply _ _ ((((@app_1 CFHeaps.loc) int) ml_array_length)...
    (** goes on for about 100 more lines *)

End Dijkstra.
```

characteristic formula

→ Axioms are justified by the soundness theorem

Verification of functors

$$\begin{tikzcd}
\text{Pqueue : PqueueSig} & \text{PqueueVerif : PqueueSpec} \arrow[l] \\
\text{Dijkstra} \arrow[u] & \text{DijkstraVerif} \arrow[l] \arrow[u]
\end{tikzcd}$$

→ **Modular verification of modular code**

Shortest path specification

```
Theorem dijkstra_spec : ∀ g x y G,
  nonnegative_edges G ->
  x \in nodes G ->
  y \in nodes G ->
  (App dijkstra g x y)
     (g ~> GraphAdjList G)
     (fun d => [d = dist G x y]
               \* g ~> GraphAdjList G)
```

mathematical graph

pre-condition

post-condition

→ Not very far from an informal specification: can be understood without knowledge of Coq

Main invariant

```
Definition hinv Q B V : hprop :=
   g ~> GraphAdjList G  (* G : graph int *)
\* v ~> Array V         (* V : array bool *)
\* b ~> Array B         (* B : array intbar *)
\* q ~> Pqueue Q        (* Q : multiset(int*int) *)
\* [inv Q B V].

Record inv Q B V : Prop := {
 Bdist: ∀x, x \in nodes G -> V\(x) = true ->
          B\(x) = dist G s x;
 Bbest: ∀x, x \in nodes G -> V\(x) = false ->
          B\(x) = mininf weight (crossing V x);
 Qcorr: ∀x, (x,d) \in Q ->
          x \in nodes G /\ ∃p, crossing V x p /\ weight p = d;
 Qcomp: ∀x p, x \in nodes G -> crossing V x p ->
          ∃d, (x,d) \in Q /\ d <= weight p;
 SizeV: length V = n;
 sizeB: length B = n }
```

Main lemma about invariant

```
Lemma inv_update : forall L V B Q x y
  x \in nodes G ->
  has_edge G x y w ->
  dy = dx + w ->
  Finite dx = dist G s x ->
  inv (V\(x:=true)) B Q (new_crossing
  If len_gt (B\(y)) dy
    then inv (V\(x:=true)) (B\(y:=Finite dy)) (\{(y, dy)} \u Q) ...
    else inv (V\(x:=true)) B Q (new_crossing x ((y,w)::L) V) .
Proof.
introv Nx Ed Dy Eq [Inv SV SB]. sets_
lets NegP: nonneg_edges_to_path Neg.
intros z. lets [Bd Bb Hc Hk]: Inv z.
(* case z = y *)
forwards~ (px&Px&Wx&Mx): (@mininf_fin
lets Ny: (has_edge_in_nodes_r Ed).
sets p: ((x,y,w)::px).
asserts W: (weight p = dy). subst p.
tests (V'\(y)) as C; case_If as Nlt.
(* subcase y visited, distance impro
false. rewrite~ Bd in Nlt. forwards M
 rewrite weight_cons in M. math.
(* subcase y visited, distance not improved *)
...
```

no reference to CF

maths-style reasoning in terms of multisets

All the nontrivial reasoning is there

180 lines of proofs in total for the invariant (a third in this lemma)

8 seconds to check

Verification of the code

```
Theorem dijkstra_spec : ∀ g x y G, ... (App dijkstra
Proof.
xcf. introv Pos Ns De. unfold GraphAdjList at 1. hdata_simpl.
xextract as N Neg Adj. xapp. intros Ln. rewrite <- Ln in Neg.
xapps. xapps. xapps. xapps*. xapps.
set (data := fun B V Q => g ~> Array N \*
  v ~> Array V \* b ~> Array B \* q ~> Heap Q).
set (hinv := fun VQ => let '(V,Q) := VQ in
 Hexists B, data B V Q \* [inv G n s V B Q (crossing
xseq (# Hexists V, hinv (V,\{})).
set (W := lexico2 (binary_map (count (= true)) (upto n))
                  (binary_map card (downto 0))).
xwhile_inv W hinv.
(* -- initial state satisfies the invariant -- *)
refine (ex_intro' (_,_)). unfold hinv,data. hsimpl.
 applys_eq~ inv_start 2. permut_simpl.
(* -- verification of the loop -- *)
intros [V Q]. unfold                                     le
(* ---- loop conditio
unfold data. xapps. x
(* ---- loop body --  )
...
Qed.
```

x-tactics

invariants

termination

lemma application

40 lines of proofs +
8 lines of invariants

15 seconds
to check

Example of a proof obligation

```
Pos : nonnegative_edges G
Ns : s \in nodes G
Ne : e \in nodes G
Neg : nodes_index G n
Adj : forall x y w : int,
      x \in nodes G -> Mem (y, w) (N\(x)) = has_edge G x y w
Nx : x \in nodes G
Vx : ~ V\(x)
Dx : Finite dx = dist G s x
Inv : inv G n s V' B Q (new_crossing G s x L' V)
EQ : N\(x) = rev L' ++ (y, w) :: L
Ew : has_edge G x y w
Ny : y \in nodes G
______________________________________(1/6)
(Let dy := Ret dx + w in
  Let _x38 := App ml_array_get b y ; in
   If_ Match
       (Case _x38 = Finite d [d] Then Ret (dy '< d) Else
       (Case _x38 = Infinite Then Ret true Else Done))
   Then (App ml_array_set b y (Finite dy) ;) ;;
         App push (y, dy) h ; Else (Ret tt))
(q ~> Pqueue Q \* b ~> Array B \* v ~> Array V' \* g ~> Array N)
(fun _:unit => hinv' L)
```

well-named hypotheses

char. formula

pre-condition

post-condition

Purely functional data structures

Trees: unbalanced, red-black

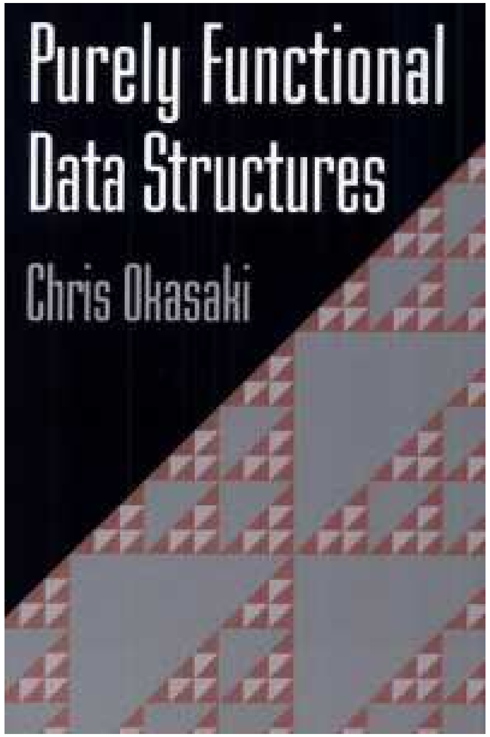
Heaps: splay, leftist, binomial, pairing

Queues: batched, lazy, realtime,
bootstrapped, HoodMelville

Dequeues: bankers

Lists: concatenable, random access

…



→ **proofs ≈ code + spec + invariants** (in nb. of lines)

→ *Program Verification Through Characteristic formulae*
(ICFP 2010)

Verified imperative programs

Algorithms and data structures:

– dijsktra's shortest path

– mutable lists (C-style lists)

– union-find (implements a partial equivalence relation)

– sparse arrays (arrays without initialization overhead)

Interaction between effects and functions:

– higher-order iterators on mutable structures (iter)

– closure with private local state (counter function)

– CPS functions (Reynold's CPS-append challenge)

– recursion through the store (Landin's knot)

→ *Characteristic formulae for the Verification of Imperative Programs* (ICFP 2011)

- Introduction
- Theory: construction of CF
- Practice: Dijkstra's algorithm
- CF in the design space
- Conclusion

Interpreting the theorem

"This piece of code admits that specification"

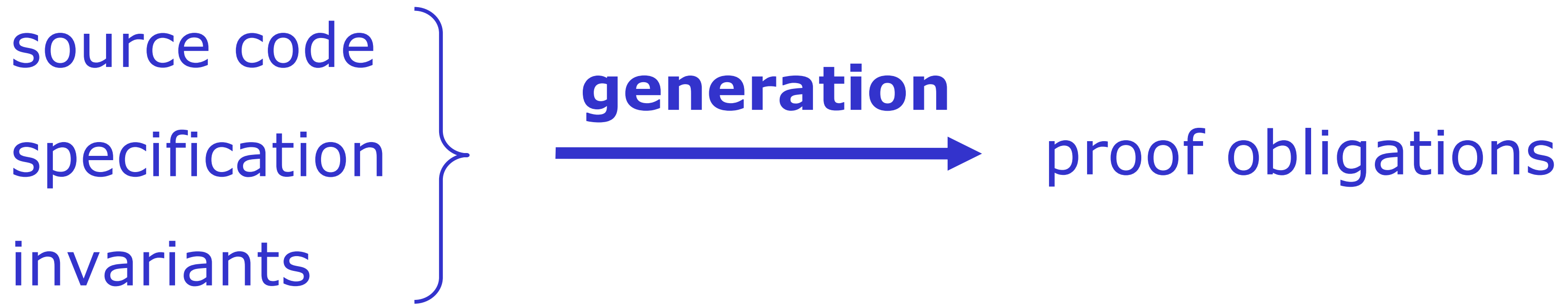
How to state and prove such a theorem?

→ A problem studied over the past 50 years

→ Five main approches, summarized next

1–Verification Condition Generators

In the traditional "Verification Condition Generator" approach, no correctness theorem is stated explicitly



source code

specification

invariants

generation

proof obligations

→ **Quite effective when proofs can be automated**

→ **If not, need more invariants (but it takes time)**

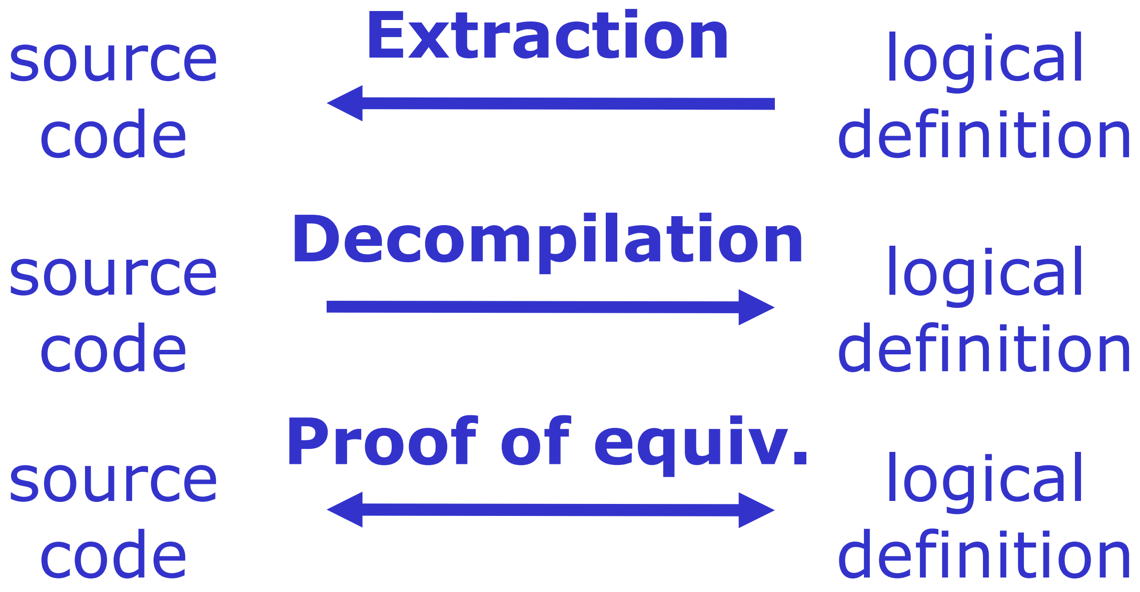
→ **or need a proof assistant (but obligations are not so easy to read and not robust on change)**

(Examples of modern VCGs: Why, Boogie, Jahob, VCC)

2– Shallow embeddings

"this logical definition admits this specification"

Three ways to relate the logical definition to the code



source code ← **Extraction** ← logical definition — e.g. Compcert, Ynot

source code → **Decompilation** → logical definition — e.g. Loop, Myreen's PhD

source code ↔ **Proof of equiv.** ↔ logical definition — e.g. Sel4

→ Large-scale projects successfully formalized

→ Partial functions and side-effects need to be encapsulated in a monad (like in Haskell code)

3– Dynamic logics

Create new mathematical logics in which the statement

"This piece of code admits that specification"

has a meaning.

Example: the Key tool, and other dynamic logics

→ **Need to build a new proof assistant: overwhelming implementation effort**

→ **Custom tool using custom logic: less trustworthy than a standard proof assistant**

4– Deep embeddings

"This piece of syntax, when executed according to such reduction rules, admits that specification"

e.g. Mehta & Nipkow, Shao et al, etc...

→ During the 2nd year of my PhD, I built a deep embedding of the pure fragment of Caml in Coq

→ **Very expressive: can prove any true property**

→ **Far from perfect: the explicit representation of syntax exposes many technical details**

→ **Characteristic formulae can be viewed as an abstract layer built on top of a deep embedding, keeping the expressiveness but hiding the details**

5– Characteristic formulae

"the characteristic formula of this piece of code is a predicate that holds of such specification"

Origins of Characteristic Formulae:

– Hennessy-Milner logic (1980): two processes are bisimilar iff their characteristic formulae are equivalent

– Honda, Berger & Yoshida (2004,2006): one can build a most-general specification (i.e. Hoare triple) of any PCF program, without referring to a representation of syntax. (Specifications expressed in an ad-hoc logic.)

Characteristic formulae in this work

1) CF expressed in a standard higher-order logic
→ accomodates a standard proof assistant

2) CF with Separation Logic style specification
→ supports modular verification

3) CF of linear size and easy to read
→ allows the approach to scale up

4) Implementation of a CF generator
→ supports verification of real Caml code

- Introduction
- Theory: construction of CF
- Practice: Dijkstra's algorithm
- CF in the design space
- ➡ **Conclusion**
  - summary
  - future work

Conclusion

– **A new, pratical approach** to program verification

– **Soundness** and **completeness** proofs

– **Implementation**: CFML, from Caml to Coq

– **Examples**: verification can be achieved at fairly reasonable cost even for complex algorithms

→ **Thesis:** generating the characteristic formula of a program and exploiting that formula in an interactive proof assistant provides an effective approach to proving that the program satisfies its specification

Future work

Direct extensions:

– support integer and real number arithmetic

– support catchable exceptions

Additional reasoning rules:

– complexity analysis (time credits)

– hidden state (anti-frame rule)

– concurrency (shared invariants)

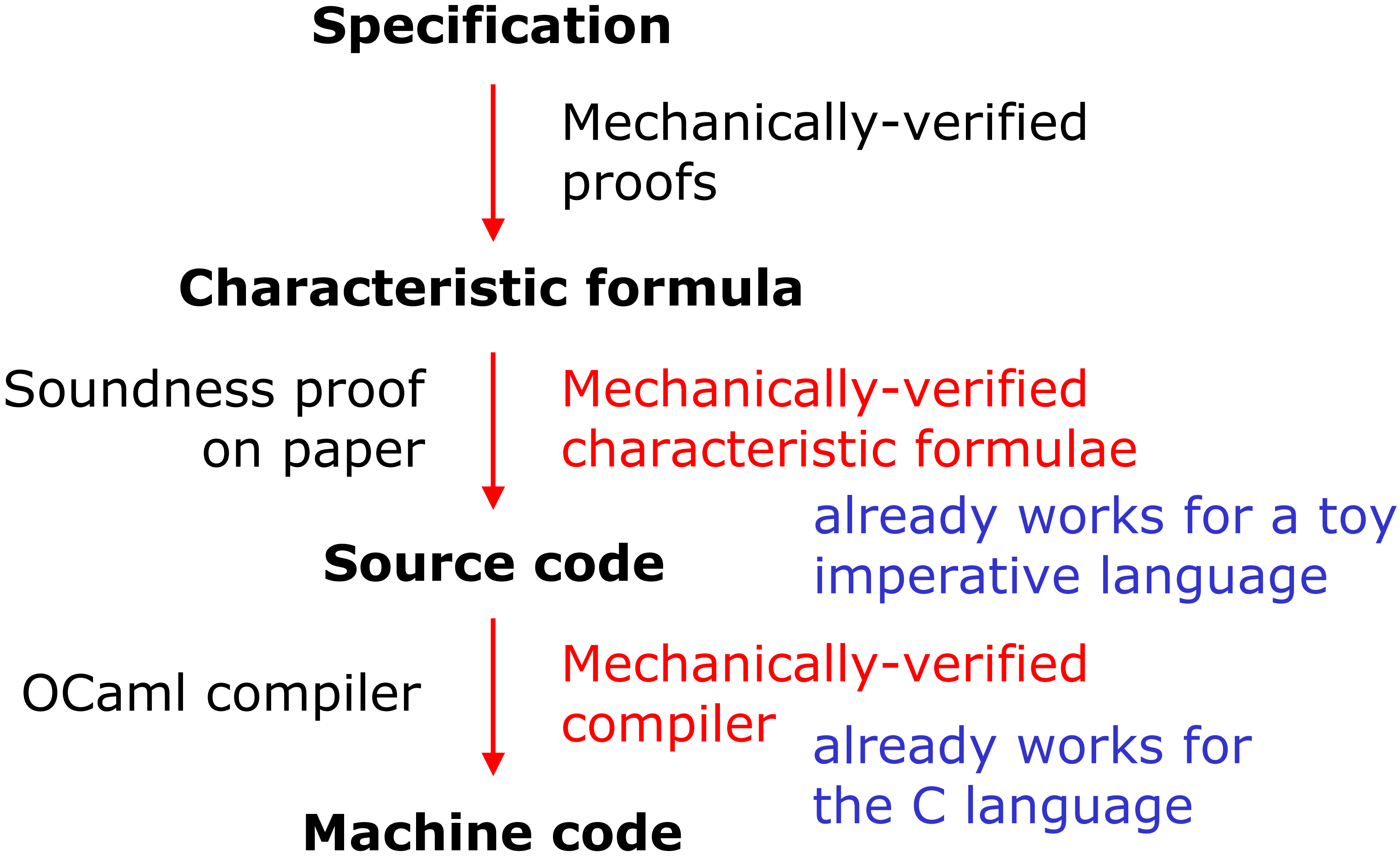
Other languages as target:

– probabilistic and cryptographic algorithms

– low-level languages (C or assembly)

– object-oriented languages (e.g., Java)

Towards a fully-verified chain



Specification

Mechanically-verified proofs

Characteristic formula

Soundness proof on paper

Mechanically-verified characteristic formulae

already works for a toy imperative language

Source code

OCaml compiler

Mechanically-verified compiler

already works for the C language

Machine code

The end!

Further information and examples: http://arthur.chargueraud.org/