The Category-theoretic Solution of Recursive Ultra-metric Space Equations

Amin Timany
iminds-DistriNet – KU Leuven
firstname.lastname@cs.kuleuven.be

Bart Jacobs
iminds-DistriNet – KU Leuven
firstname.lastname@cs.kuleuven.be

Abstract
We give a short description of our implementation in Coq supporting the construction of category-theoretic solutions to recursive ultra-metric space equations for domain theory. This is one step in our efforts to provide a category-theoretical foundation for program semantics and program logics.

Introduction
One particular difficulty in defining a denotational semantics for concurrent higher-order imperative programming languages is the fact that their models are solutions to recursive and sometimes circular equations. The same applies to defining languages with higher-order store we need a model of the program’s semantics is usually used to reason about soundness of such program semantics and program logics.

Solution in M-categories
Such recursive and circular domain-theoretic equations are usually solved in a category enriched over a category with extra structure which allows construction of such solutions. These solutions are usually in the form of the fixed points of some functors unique up to isomorphism. Such a method is presented in [1] and compared to some other relevant works.

In [1], the extra structure is that of a non-empty complete bounded ultra-metric space. An ultra-metric space consists of a set $M$ and a distance function $\delta : M \times M \to \mathbb{R}^+$ to the positive real numbers such that:

**UM-1** $\forall x, y. \delta(x, y) = 0 \iff x = y$

**UM-2** $\forall x, y. \delta(x, y) = \delta(y, x)$

**UM-3** $\forall x, y, z. \delta(x, y) \leq \max(\delta(x, z), \delta(y, z))$

An ultra-metric space is complete if every Cauchy sequence has a limit. It is bounded if the codomain of $\delta$ instead of $\mathbb{R}^+$ is the set $[0, b]$ for some $b \in \mathbb{R}^+$. A function $f : M \to M'$ from one ultra metric space to another is called non-expansive if $\delta'(f(x), f(y)) \leq \delta(x, y)$ and contractive if there is a $c < 1$ such that $\delta'(f(x), f(y)) \leq c \cdot \delta(x, y)$.

Intuitively, the distance function of an ultra metric space can be thought of as the degree of similarity of two elements rather than their spatial distance. One particular class of ultra-metric spaces are bisected ultra-metric spaces where the distance function $\delta : M \times M \to \{0\} \cup \{2^{-n} | n \in \mathbb{N}\}$. As an instance, consider the space of functions from natural numbers to a set $A$ where the distance is defined as:

$$\delta(f, g) = \begin{cases} 0 & \text{if } f = g \\ 2^{\max\{n | \forall m < n. f(m) = g(m)\}} & \text{otherwise} \end{cases}$$

In [1], the authors call a category $\mathcal{C}$ an M-category if it is enriched over the category of non-empty complete 1-bounded ultra-metric spaces $\text{CBULT}_{ne}$. It is furthermore required that the composition operation for morphisms of $\mathcal{C}$ forms a non-expansive function. The domain of this composition function is taken to be the product of the ultra-metric spaces in the usual sense (the product in $\text{CBULT}_{ne}$).

A functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ from one M-category to another is called locally non-expansive and locally contractive if its morphism maps are respectively non-expansive and contractive. In [1], the authors show that any mixed-variance locally-contractive functor $\mathcal{F} : \text{C}^{\text{op}} \times \mathcal{C} \to \mathcal{C}$ has a unique fixed point up to isomorphism whenever $\mathcal{C}$ has a terminal object and limits (category theoretical notion of limit) of some class of functors which they call increasing Cauchy towers. An object $A$ is a fixed point of $\mathcal{F}$ if we have $\mathcal{F}(A, A) \simeq A$.

Implementation
We have implemented (see [5]) the theory of construction of solutions to locally-contractive functors of [1] in Coq. This implementation is based on our general implementation of category theory [4]. In what follows we describe this development and give a brief comparison between this work and the other very recent independent development implementing this theory [3].

Ultra-metric spaces
In order to avoid working with real numbers and having to deal with their idiosyncrasies, we have developed a more general notion of ultra-metric spaces. That is, instead of real numbers we use what we call an M-lattice. An M-lattice is a preorder relation $(X, \sqsubseteq)$ such that

**ML-1** Has a bottom element $\bot$

**ML-2** Has meet $(\sqcap)$ of arbitrary subsets of $X$

**ML-3** Has a top element $\top$

**ML-4** $\text{Appr}(X)$ is a subset of $X$ of approximation elements

**ML-5** $\forall a. a \in \text{Appr}(X) \to \bot \sqsubseteq a$

**ML-6** $\forall a. \bot \sqsubseteq a \to \exists b \in \text{Appr}(X), b \sqsubseteq a$

**ML-7** $\forall a. (\forall b \in \text{Appr}(X). a \sqsubseteq b \to a = \bot)$

**ML-8** $\forall a \in \text{Appr}(X), \exists b \in \text{Appr}(X), b \sqsubseteq a \lor (\exists c \in \text{Appr}(X), \forall a. a \sqsubseteq c \to a = \bot)$
Conditions $\text{ML-1}$ and $\text{ML-2}$ imply that $X$ is a complete meet-lattice (in the order-theoretic sense). The elements $\bot$ and $\top$ respectively play the role of $0$ and $1$ (the bound).

In practice we only care about approximations (e.g., of limits) only for distances in $\text{Appr}(X)$. For instance, one can work with real numbers but only care for approximation of limits only up to rational numbers. The rest of the conditions are to allow us to prove that defining limits with approximations up to approximation elements have the desired properties, e.g., uniqueness and Banach's fixed point theorem. The disjunction and existential quantifiers in Condition $\text{ML-3}$ are respectively represented as sum types and $\Sigma$ types (dependent sum type) to allow their elimination in computational contexts.

The notion of $\text{ML}$-lattice as described above allows us to develop a general theory of ultra-metric spaces. They allow us to prove general properties required, e.g., the fact that $\text{CBUILT}_{ne}$ itself forms a complete cartesian-closed $\text{M}$-category. In $\text{M}$, the authors only provide support for bisected ultra-metric spaces. We represent bisected spaces by providing an $\text{M}$-lattice whose elements have the desired properties, e.g., uniqueness and Banach's fixed point theorem. The disjunction and existential quantifiers in Condition $\text{ML-3}$ are respectively represented as sum types and $\Sigma$ types (dependent sum type) to allow their elimination in computational contexts.

Fixed points We prove the existence of fixed points of mixed variance locally contractive functors the same way as in $\text{M}$. The proof of uniqueness of fixed points in our development is slightly different. This is due to our different requirements, i.e., lack of non-emptiness condition. Therefore, we prove that the fixed point of $\mathcal{F} : \text{Coop} \times \mathcal{C} \to \mathcal{C}$ constructed is unique in the full subcategory of $\mathcal{C}$ where every object $A$ has a morphism $f_A : 1 \to A$. This subcategory is indeed an $\text{M}$-category in which any morphism set is non-empty. We show that the fixed point constructed is in this subcategory and furthermore that any two fixed points of $\mathcal{F}$ in this subcategory are isomorphic. The implementation of $\text{M}$ does not provide any proof of uniqueness. We use all the necessary category-theoretical concepts and lemmas, e.g., (co)limits, their uniqueness up to isomorphism, etc., that are necessary to prove existence and uniqueness of the fixed points from $\text{M}$.

Use of axioms Our use of axioms is not limited to the axioms of propositional and functional extensionality mentioned earlier. We have also had to made use of some other axioms. Namely, we have used the axiom of constructive indefinite description on the natural numbers and not_all_ex_not from the library of Coq to prove Condition $\text{ML-4}$ for bisected spaces. This is in turn only used to show that whenever two sequence have their pointwise distances less than some distance is non-expansive which is required for the proof that $\text{CBUILT}$ (we don't construct $\text{CBUILT}_{ne}$) has all exponentials. In $\text{M}$, by using setoids and considering only bisected spaces the authors require no axioms to prove similar results.

References


